

Rotation of trihedron along the curve

A C curve is given by parametric equation $\mathbf{r}=\mathbf{r}(u)$ (in vector view). In every point, which is on the C curve, vectors tangent \mathbf{t} , normal \mathbf{n} and binormal \mathbf{b} are determined (as column-vectors). These vectors form trihedron.

However, in singularity point $|\mathbf{t}|=0$ and to determine trihedron correctly if standard methods are used.

Following method used in new algorithm.

Trihedron $\{\mathbf{t}^*, \mathbf{n}^*, \mathbf{b}^*\}$ computes in point which is next to singularity point. In singularity point tangent vector \mathbf{t} computed (by Taylor-series or by three-point approximation). For searching \mathbf{n} and \mathbf{b} vectors we rotate the trihedron $\{\mathbf{t}^*, \mathbf{n}^*, \mathbf{b}^*\}$ to coincide of \mathbf{t} and \mathbf{t}^* vectors. Vectors \mathbf{n} and \mathbf{b} ensue as result of rotation \mathbf{n}^* and \mathbf{b}^* accordingly.

Trihedron turns around axe $\vec{p} = \frac{\begin{bmatrix} \vec{t}^* \\ \vec{t} \end{bmatrix}}{\left\| \begin{bmatrix} \vec{t}^* \\ \vec{t} \end{bmatrix} \right\|}$ (i.e. $\|\vec{p}\| \equiv 1$). Rotation angle is angle between \mathbf{t} and \mathbf{t}^* vectors (let

us use the symbol γ for this angle).

Rotating an \mathbf{t}^* vector is combining of following transformations:

1. Coinciding of \mathbf{p} - and \mathbf{Z} -axes:

1.1. Rotate \mathbf{p} around X-axes (to coincide of \mathbf{p} and \mathbf{XOZ} -plane). Rotation angle is α , where

$$\cos \alpha = \frac{p_z}{\sqrt{p_y^2 + p_z^2}}, \quad \sin \alpha = \frac{p_y}{\sqrt{p_y^2 + p_z^2}}. \quad \text{Rotation matrix is}$$

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}$$

1.2. Rotate $T_1 \vec{p}$ around Y-axes. Rotation angle is β , where $\cos \beta = \frac{p_x}{\sqrt{p_y^2 + p_z^2}}$, $\sin \beta = p_x$. Rotation matrix is

$$T_2 = \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{pmatrix}$$

2. Rotating an $T_2 T_1 \vec{t}^*$ around \mathbf{Z} -axe to γ angle. Rotation matrix is

$$T_3 = \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. Return vector \mathbf{p} to previous position. Rotation matrix is

$$T_4 = T_1^{-1} T_2^{-1}.$$

Summarize transform matrix is

$$T = T_4 T_3 T_2 T_1.$$

New tangent vector is $T \cdot \vec{t}^*$, new normal vector is $T \cdot \vec{n}^*$, new binormal vector is $T \cdot \vec{b}^*$.

After simplifying T matrix may rewritten as

$$T = \begin{pmatrix} (1 - \cos \gamma)p_x^2 + \cos \gamma & (1 - \cos \gamma)p_x p_y - p_z \cdot \sin \gamma & (1 - \cos \gamma)p_x p_z + p_y \cdot \sin \gamma \\ (1 - \cos \gamma)p_x p_y + p_z \cdot \sin \gamma & (1 - \cos \gamma)p_y^2 + \cos \gamma & (1 - \cos \gamma)p_y p_z - p_x \cdot \sin \gamma \\ (1 - \cos \gamma)p_x p_z - p_y \cdot \sin \gamma & (1 - \cos \gamma)p_y p_z + p_x \cdot \sin \gamma & (1 - \cos \gamma)p_z^2 + \cos \gamma \end{pmatrix}$$